[12.16] (Observe the use of the Einstein Summation Convention in this problem). Let **M** be an *n*-manifold, *q* ≤ *n*, and M =  {1, 2,…, *n*}. Let ****** be a *p*-form  and let ****** be its *q*-vector Hodge-like dual .

1. Confirm the equivalency of the 3 conditions for simplicity:
   1. ****** is simple iff  for all *p*-tuples (*r*, …, *t*) and (*r*’, …, *t*’) in M *p*
   2. ****** is simple iff  for all *q*-tuples (*u*, …, *w*) and (*u*’, …, *w*’) in M *q*
   3. ****** and ****** are both simple iff  for all *p*-tuples (*w*, *s* …, *t*) and *q*‑tuples (*u*, …, *w*).
2. Show that a *2*-form in **M** is simple iff  for all pairs (*r*, *s*) and (*r*’, *s*’) in M 2

Proof: Einstein summation convention is used throughout this proof.

This is a rework for myself of Juergen Beckmann’s proof with additional details and different wording so that I can more easily follow it. It also isimplifies his part (b) proof by skipping the vectors that Penrose mentions in his hint.

(a)

Preface: I find it easier to use equations rather than proportions for the Hodge-like duals ****** and ******. The most logical choice is to make both proportionality constants the same,  , as in

 and

 where **∈** is an *n*‑vector such that 

To see that this works, set the constant to  in (ii). We then see that (i) must have the same constant [due to equation (iii) ] since



Notice that the indices used in the *p*-tuple in (3) are slightly different than in (1) and (2). Because in this proof we will be switching often between the expressions in (1), (2), and (3) we require a set of non-conflicting indices. To do this we re-write (i) and (ii):

where *r*…*t* are given distinct members of M and

*x*…*z* sums over the remaining members of M, and

 where *u*…*w* are given distinct members of M and

*a*…*c* sums over the remaining members of M.

We also must keep in mind the following fact:

(vi) Each set *r*…*t x*…*z* and *a*…*c u*…*w* is a permutation of the members of M = {1, 2,…, *n*}.

To prove part (a) it is sufficient to show that  iff  iff  since then we will have ****** is simple iff  iff  iff ****** is simple iff  iff ****** and ****** are both simple.



Therefore  iff . ✔



Therefore  iff  ✔

(b).

Recall that ****** is antisymmetric in (*r*, *s*) :

****** is a sum of, say, *q* simple 2-forms. So there are simple 1-forms *******k* and *******k* such that 

That is,

(\*) 

Thus ****** is antisymmetric in (*r*, *s*). ✔

****** is simple iff ∃ 1-forms That is, ****** is simple iff the components *r s*of satisfy

”IF”

Suppose for all pairs (*r*, *s*) and (*r*’, *s’*) in M 2.

If ****** = 0, clearly ****** (e.g., ****** *dx*1 ∧ *dx*1). So, suppose ****** ≠ 0. Then ∃ *a*, *b* ∈{ 1, …, *n* } such that *a b* ≠ 0. Define



Consequently 

(iii) 

Now 

(iv) 

Plugging (iv) into (iii) yields





⇒ ******is simple ✔

“ONLY IF”

Suppose  is simple. Then ∃ 1‑forms  such that

(v) 

So

