[12.16] Let **M** be an *n*-manifold, *n* = *p* + *q*, and M =  {1, 2,…, *n*}. Let ****** be a *p*-form  and let ****** be its *q*-vector Hodge-like dual .

1. Confirm the equivalency of these 3 conditions for simplicity:
   1. ****** is simple iff  for all *p*-tuples (*r*, …, *t*) and (*r*’, …, *t*’) in M*p*
   2. ****** is simple iff  for all *q*-tuples (*u*,…,*w*) and (*u*’,…,*w*’) in M*q*
   3. ****** and ****** are both simple iff  for all *p*-tuples (*w*, *s* …, *t*) and *q*‑tuples (*u*, …, *w*).
2. Show that a *2*-form in **M** is simple iff  for all pairs (*r*, *s*) and (*r*’, *s*’) in M 2

Proof: Einstein summation convention is used throughout this proof. This is a re-write of Juergen Beckmann’s proof so that I can better understand it. I identify a hole in his proof of part (a) and I simplify his proof of part (b).

****** is the Hodge-like dual of ******means  or, equivalently, . Recall





Let *K* and *L* be the proportionality constants. Then



A reasonable choice is to set *K* = *L* = . So the Hodge-like dual definitions become



(ii) 

(iii) 

The definitions of Simple are:

****** is simple means  such that . That is,

(iv) 

****** is simple means  such that . That is,

(v) 

(a)

To prove  one must show  and . Thus, to prove (a) we must prove the following two assertions.

(4) ****** is simple iff its Hodge-like dual ****** is simple

(5)  iff  iff 

Note: In his proof, Juergen Beckmann did not include (4). Rather he claimed that (5) plus Part (b) above (his Part a) is sufficient. But Part (b) is merely an example of proving (1) for a special case. I don’t see that (1) + (5) can result in ****** being simple. I believe (4) + (5) is required.

Proof of (4)

****** is simple  such that .

That is, such that

.

Note that if any subscript repeats then that term is zero. Thus we only consider permutations (*r*, …, *t*) of (1, …,*p*) in the sum.

Its Hodge-like dual ****** has components

 = 

where  is the permutation  and sign() is +1 if the permutation is even, -1 if it is odd.

We need to define *q* 1-vector fields  such that

.

That is, such that

.

Again, we only need consider permutations of (*u*, *v*, …, *w*) of (*p*+1, *p*+2, …, *n*) in the sum. Thus, (*r*, …, *t*, *u*, *v*, …, *w*) is a permutation of (1, 2, …, *n*).

So, we wish to define the *q* 1-vector fields so that

.

Because there is not a 1-1 matchup of vectors and 1-forms, we cannot make a simple definition like . Rather, to define the *q* components of ****** in terms of the *p* components of ****** we define *xu* to be the entire RHS (and we have to toss in a *q*! factor) and we set the other components to either 0 or 1 as required:

, ,

.

Note: Since the term  includes , we must define them all.

Finally, we check that these definitions give the desired answer:







Proof of (5)

By renaming subscripts, (ii) and (iii) can also be expressed as

where *ws*…*t* are *p* given distinct members of M and *x*…*z* are the remaining *q* members of M, and

 where *u*…*w* are *q* distinct members of M and

*a*…*c* are the remaining *p* members of M.

We also must keep in mind that each set *r*…*t x*…*z* and *a*…*c u*…*w* is a permutation of the members of M = {1, 2,…, *n*}. There is no need to consider terms with duplicate indices since .

Let **P** be the set of permutations of (*r*, … , *t*, *w*). Then



Therefore  iff . ✔

Let **P** \* be the set of permutations of (*x*, … , *z*, *w*). Then



Therefore  iff  ✔

(b). This is a simplification of Juergen Beckmann’s proof (he calls this his part a). The simplification occurs in the “IF” part of the proof by eliminating Beckmann’s use of the vectors that Penrose mentions in his hint for this problem.

Recall that ****** is antisymmetric in (*r*, *s*) :

(\*) 

****** is simple iff ∃ 1-forms That is,

(\*\*) ****** is simple iff ∃ 1-forms ****** and ****** such that the components *r s*of  satisfy



”IF”

Suppose for all pairs (*r*, *s*) and (*r*’, *s’*) in M 2.

If ****** = 0, clearly ****** is simple (e.g., ****** *dx*1 ∧ *dx*1). So, suppose ****** ≠ 0. Then

∃ *a*, *b* ∈ { 1, …, *n* } such that . Fix (*r*, *s*) ∈ M2. Define



We will show that . First,



So,

(III) 

Now 

(IV) 

Plugging (IV) into (III) yields



 ⇒ ******is simple [due to (\*\*)] ✔

“ONLY IF”

Suppose  is simple. Then from (\*\*) ∃ 1‑forms  such that

(V) 

So

